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Jauch–Piron States on Concrete Quantum Logics

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We exhibit an example of a concrete (=set-representable) quantum logic which is not a Boolean algebra such that every state on it is Jauch–Piron. This gives a negative answer to a problem raised by Navara and Pták. Further we show that such an example does not exist in the class of complete (i.e., closed under arbitrary disjoint unions) concrete logics.

A *concrete logic* is a pair $\langle X, \mathscr{L}\rangle$ where X is a set and $\mathscr{L} \subset \exp X$ satisfies:

1. $\emptyset \in \mathscr{L}$.
2. $X - A \in \mathscr{L}$ whenever $A \in \mathscr{L}$.
3. $A \cup B \in \mathscr{L}$ whenever $A, B \in \mathscr{L}$, $A \cap B = \emptyset$.

Note that also $A - B = X - (B \cup (X - A)) \in \mathscr{L}$ for every $A, B \in \mathscr{L}$, $B \subset A$. A state on a concrete logic $\langle X, \mathscr{L}\rangle$ is a mapping $s: \mathscr{L} \to [0, 1]$ such that:

1. $s(A \cup B) = s(A) + s(B)$ whenever $A, B \in \mathscr{L}$, $A \cap B = \emptyset$.
2. $s(X) = 1$.

A state s is said to be *carried by a point* $x \in X$ if $s(A) = 1$ whenever $x \in A$, $A \in \mathscr{L}$, and $s(A) = 0$ if $x \notin A$. A state s is called *Jauch–Piron* if, for every $A, B \in \mathscr{L}$ with $s(A) = s(B) = 1$, there exists a $C \in \mathscr{L}$ such that $C \subset A \cap B$ and $s(C) = 1$.

The Jauch–Piron property of states has been considered as physically natural in the axiomatization of quantum systems (Jauch, 1968; Jauch and Piron, 1969; Piron, 1976) and further studied by a number of authors (see, e.g., Amman, 1987; Bugajska and Bugajski, 1972; Bunce *et al.*, 1985; Pták and Pulmannová, 1991). Navara and Pták (1989) (see also Müller *et al.*, 1993)

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raised the question of whether a concrete logic such that every state on it is Jauch–Piron is necessarily a Boolean algebra. We give a negative answer to this question.

Theorem 1. There exists a concrete logic $\langle X, \mathscr{L}\rangle$ which is not a Boolean algebra such that every state on $\langle X, \mathscr{L}\rangle$ is Jauch–Piron.

Proof. Let K_1, K_2 be two disjoint sets of cardinality ω_1. Let $K = K_1 \cup K_2$ and $X_n = K^n$ $(n = 1, 2, \ldots)$. Further, let

$$Q_n = \{\langle k_1, \ldots, k_n\rangle \in X_n, k_i \in K_2 \text{ for an even number of indices } i\}$$

and $Q'_n = X_n - Q_n$. For $1 \le m \le n$ denote by $\pi_{n,m}: X_n \to X_m$ the projection onto the first m coordinates, i.e., $\pi_{n,m}(\langle k_1, \ldots, k_n\rangle) = \langle k_1, \ldots, k_m\rangle$ for all $\langle k_1, \ldots, k_n\rangle \in X_n$.

We define by induction on n a system $\mathscr{L}_n \subset \exp X_n$ $(n = 1, 2, \ldots)$. For $n = 1$ and $M \subset X_1 = K$ let $M \in \mathscr{L}_1$ if and only if either M is finite and $|M \cap K_1| = |M \cap K_2|$ or $X_1 - M$ is finite and $|(X_1 - M) \cap K_1| = |(X_1 - M) \cap K_2|$.

Suppose $\mathscr{L}_n \subset \exp X_n$ is already defined. We define $\mathscr{L}_{n+1}$ as the family of all sets $M \subset X_{n+1}$ which can be expressed as a disjoint union of the form

$$M = R \times K \cup \bigcup_{j=1}^{m} (\{p_j\} \times G_j) \cup F \tag{1}$$

where $R \in \mathscr{L}_n$, $p_j \in X_n$ $(j = 1, \ldots, m)$, $m < \infty$, $G_j \subset K$, $K - G_j$ is a finite set and $|K_1 - G_j| = |K_2 - G_j| + 1$ for $j = 1, \ldots, m$, and $F \subset X_{n+1}$ is a finite set satisfying $|F \cap Q_{n+1}| = |F \cap Q'_{n+1}|$.

Notice that if $M \in \mathscr{L}_{n+1}$ is infinite (i.e., if either $R \neq \varnothing$ or $m \neq 0$), then $M \cap Q_{n+1}$ and $M \cap Q'_{n+1}$ are infinite as well.

I. If $M, M' \in \mathscr{L}_n$ and $M \cap M' = \varnothing$, then $M \cup M' \in \mathscr{L}_n$. This is clear for $n = 1$. For $n \ge 2$ it follows easily from (1) by induction on n.

II. If $M \in \mathscr{L}_n$, then $X_n - M \in \mathscr{L}_n$, i.e., $\langle X_n, \mathscr{L}_n\rangle$ is a concrete logic.

Proof. By induction on n: Let $M \in \mathscr{L}_{n+1}$ be of the form (1). Denote

$$Z = \bigcup_{j=1}^{m} (\{p_j\} \times G_j) \cup F, \qquad \text{i.e.,} \quad M = R \times K \cup Z$$

Clearly $\pi_{n+1,n}(Z)$ is a finite set and $\pi_{n+1,n}(Z) \cap R = \varnothing$. We prove that $X_{n+1} - M \in \mathscr{L}_{n+1}$ in several steps:

(a) There exists a finite set $R' \in \mathscr{L}_n$ such that $R \cap R' = \varnothing$ and $R' \supset \pi_{n+1,n}(Z)$. This is clear if $X_n - R$ is finite: we can take $R' = X_n - R \in \mathscr{L}_n$ by the induction assumption. If $X_n - R$ is infinite, then also $(X_n - R) \cap Q_n$ and

$(X_n - R) \cap Q'_n$ are infinite and it is easy to see that there exists a finite set $R' \subset X_n - R$ such that $|R' \cap Q_n| = |R' \cap Q'_n|$ and $R' \supset \pi_{n+1,n}(Z)$. Then $R' \in \mathscr{L}_n$ by (1).

(b) Let R' be the set constructed in (a). Then

$$X_{n+1} - M = [(X_n - (R \cup R')) \times K] \cup (R' \times K - Z)$$

where $[X_n - (R \cup R')] \times K \in \mathscr{L}_{n+1}$ and the above union is disjoint. Thus it suffices to prove that $R' \times K - Z \in \mathscr{L}_{n+1}$.

(c) Recall that $Z = \bigcup_{j=1}^{m} (\{p_j\} \times G_j) \cup F$, where $F = \{\langle q_i, k_i \rangle, i = 1, \ldots, r\}$ and $p_j, q_i \notin R'$, $k_i \in K$ for all i, j. Clearly there exists a set $G \subset K$ such that $G \subset G_j$ $(j = 1, \ldots, m)$, $G \cap \{k_1, \ldots, k_r\} = \varnothing$, and $K - G$ is finite and satisfies $|K_1 - G| = |K_2 - G| + 1$, i.e., $|(G_j - G) \cap K_1| = |(G_j - G) \cap K_2|$ for all $j = 1, \ldots, m$. We have

$$\begin{aligned} R' \times K - Z &= (R' \times G) \cup [R' \times (K - G)] \\ &\quad - \left[(\{p_1, \ldots, p_m\} \times G) \cup \bigcup_{j=1}^{m} (\{p_j\} \times (G_j - G)) \cup F \right] \\ &= [(R' - \{p_1, \ldots, p_m\}) \times G] \cup [F_1 - (F_1 \cup F)] \end{aligned}$$

where the union is disjoint, $(R' - \{p_1, \ldots, p_m\}) \times G \in \mathscr{L}_{n+1}$ by (1), and $F_1 = R' \times (K - G)$, $F_2 = \bigcup_{j=1}^{m} [\{p_j\} \times (G_j - G)]$ are finite sets. It is sufficient to prove $F_1 - (F_2 \cup F) \in \mathscr{L}_{n+1}$.

We have

$$|F_1 \cap Q_{n+1}| = |F_1 \cap Q'_{n+1}|$$

as

$$|R' \cap Q_n| = |R' \cap Q'_n|,$$

$$|F_2 \cap Q_{n+1}| = |F_2 \cap Q'_{n+1}|$$

as

$$|(G_j - G) \cap K_1| = |(G_j - G) \cap K_2|$$

and

$$|F \cap Q_{n+1}| = |F \cap Q'_{n+1}|$$

by (1). Hence

$$|[F_1 - (F_2 \cup F)] \cap Q_{n+1}| = |[F_1 - (F_2 \cup F)] \cap Q'_{n+1}|$$

and $F_1 - (F_2 \cup F) \in \mathscr{L}_{n+1}$.

III. Denote $X=\prod_{i=1}^{\infty} K$, $\mathscr{L}\subset \exp X$,

$$\mathscr{L}=\bigcup_{n=1}^{\infty}\left\{A\times \prod_{i=n+1}^{\infty} K, A\in\mathscr{L}_n\right\}$$

By the definition of $\mathscr{L}_n$ we have $A\times K\in\mathscr{L}_{n+1}$ for every $A\in\mathscr{L}_n$, hence $\{A\times \prod_{i=n+1}^{\infty} K, A\in\mathscr{L}_n\}$ is an increasing sequence of concrete logics. Therefore also $\langle X, \mathscr{L}\rangle$ is a concrete logic.

IV. If $A, B\in\mathscr{L}_n$, then there exist sets $M_\alpha, N_\alpha\in\mathscr{L}_{n+1}$ $(\alpha<\omega_1)$ such that $M_\alpha\cup N_\alpha=(A\cap B)\times K$ for every $\alpha<\omega_1$ and $M_\alpha\cap M_\beta=\varnothing$ $(\alpha\neq\beta)$.

Proof. (a) The statement is true if there exists an $L\in\mathscr{L}_n$ such that $L\subset A\cap B$ and $(A\cap B)-L$ is finite. Indeed, we can take

$$M_\alpha=[(A\cap B)-L]\times\{x_\alpha, x'_\alpha\}$$

and

$$N_\alpha=(L\times K)\cup[((A\cap B)-L)\times(K-\{x_\alpha\})]\qquad(\alpha<\omega_1)$$

where $K_1=\{x_\alpha, \alpha<\omega_1\}$ and $K_2=\{x'_\alpha, \alpha<\omega_1\}$. Clearly M_α and N_α satisfy all the conditions required.

(b) In general we prove the statement by induction on n. For $n=1$ the statement follows from (a). Let $A, B\in\mathscr{L}_{n+1}$. Then A and B can be expressed as disjoint unions $A=\bigcup_{i=1}^{k} A_i$, $B=\bigcup_{j=1}^{l} B_j$ of the form (1). It is sufficient to prove the statement for all the pairs A_i, B_j [where these sets are of the form $R\times K$, $\{p\}\times G$, or F; see (1)]. Indeed, suppose we have found the sets $M_\alpha^{i,j}$, $N_\alpha^{i,j}$ satisfying the conditions of the statements for the pairs A_i, B_j. Then $M_\alpha=\bigcup_{i,j} M_\alpha^{i,j}$, $N_\alpha=\bigcup_{i,j} N_\alpha^{i,j}$ satisfy the conditions for the pair A, B.

We prove the statement for the sets A, B of the form $R\times K$, $\{p\}\times G$, F. If either of sets A, B is finite, we can apply (a).

If $A=R\times K$, $B=\{p\}\times G$, then either $A\cap B=\varnothing$ (if $p\notin R$) or $B\subset A$ (if $p\in R$). In both cases $A\cap B\in\mathscr{L}_{n+1}$ and we can apply (a).

Let $A=\{p\}\times G$, $B=\{p'\}\times G'$. If $p\neq p'$, then $A\cap B=\varnothing$. If $p=p'$, then we can find $G''\subset G\cap G'$ such that $\{p\}\times G''\in\mathscr{L}_{n+1}$ and $A\cap B-(\{p\}\times G'')=\{p\}\times[(G\cap G')-G'']$ is finite.

It remains the case that $A=R\times K$, $B=R'\times K$ with $R, R'\in\mathscr{L}_n$. In this case the statement follows from the induction assumption.

V. If $A, B\in\mathscr{L}$, then there exist sets M_α, $N_\alpha\in\mathscr{L}$ $(\alpha<\omega_1)$ such that $M_\alpha\cup N_\alpha=A\cap B$ $(\alpha<\omega_1)$ and $M_\alpha\cap M_\beta=\varnothing$ $(\alpha\neq\beta)$.

This follows from the definition of $\mathscr{L}$ and from IV.

VI. Every state on $\langle X, \mathscr{L}\rangle$ is Jauch–Piron.

Proof. Let $s: \mathscr{L} \to [0, 1]$ be a state. Let $A, B \in \mathscr{L}$, $s(A) = s(B) = 1$. Find the sets M_α, N_α $(\alpha < \omega_1)$ satisfying the properties of V. As the sets M_α are disjoint, $s(M_\alpha) \neq 0$ only for a countable set of indices α. Thus $s(M_\beta) = 0$ for some β. Then $s(A - M_\beta) = 1$ and $(A - M_\beta) \cap (B - N_\beta) = \varnothing$, so that $s(B - N_\beta) = 0$ and $s(N_\beta) = 1$, where $N_\beta \in \mathscr{L}$, $N_\beta \subset A \cap B$.

VII. To prove that $\langle X, \mathscr{L} \rangle$ is not a Boolean algebra, we shall use the following notations:

For $M \subset X_n$, $n \geq 1$, and $p \in X_{n-1}$ denote $M_p = \{k \in K, \langle p, k \rangle \in M\}$ (for $n = 1$ we consider formally X_0 as a one-point set). For $M \subset X_n$ and $1 \leq k < n$ define $q_{n,k}(M)$ inductively by $q_{n,n-1}(M) = \{p \in X_{n-1}, M_p \text{ is infinite}\}$ and $q_{n,k}(M) = q_{n-1,k}(q_{n,n-1}(M))$ for $1 \leq k \leq n-1$.

We say that a set $M \subset X_n$, $n \geq 1$, is *admissible* if: (1) either M_p or $K - M_p$ is finite for every $p \in X_{n-1}$, (2) $\{p \in X_{n-1}, \varnothing \neq M_p \neq K\}$ is a finite set.

Let $M \subset X_n$ $(n \geq 1)$ be an admissible set and let $p \in X_{n-1}$. We define

$$f_M(p) = \begin{cases} |(\{p\} \times M_p) \cap Q_{n+1}| - |(\{p\} \times M_p) \cap Q'_{n+1}| & \text{if } M_p \text{ is finite} \\ |(\{p\} \times (K - M_p)) \cap Q'_{n+1}| & \\ \quad - |(\{p\} \times (K - M_p)) \cap Q_{n+1}| & \text{if } |K - M_p| < \infty \end{cases}$$

We define further

$$r(M) = \sum_{p \in X_{n-1}} f_M(p)$$

(note that this sum is finite). Clearly $f_{M \cup M'}(p) = f_M(p) + f_{M'}(p)$ and $r(M \cup M') = r(M) + r(M')$ for all disjoint admissible sets $M, M' \subset X_n$ and for all $p \in X_{n-1}$.

VIII. Let $M \in \mathscr{L}_n$ $(n \geq 1)$. Then (a) M is an admissible set, (b) $q_{n,k}(M)$ is an admissible set for every k, $1 \leq k < n$, and (c)

$$r(M) + \sum_{k=1}^{n-1} r(q_{n,k}(M)) = 0$$

We prove the statement by induction on n. For $n = 1$ the statement is clear [every $M \in \mathscr{L}_1$ is admissible and $r(M) = 0$]. Suppose that the statements (a)–(c) are true for n and let $M \in \mathscr{L}_{n+1}$ be of the form (1). Clearly M is admissible. Further, $q_{n+1,n}(M) = R \cup \{p_1, \ldots, p_m\}$ and

$$q_{n+1,k}(M) = q_{n,k}(R \cup \{p_1, \ldots, p_m\}) = q_{n,k}(R)$$

for $k = 1, \ldots, n-1$. By the induction assumption the sets R and $q_{n,k}(R)$ $(k = 1, \ldots, n-1)$ are admissible and $\{p_1, \ldots, p_m\}$ is a finite set, so that also

$R \cup \{p_1, \ldots, p_m\}$ is admissible. This proves statements (a) and (b) for $n+1$. As the function r is additive, we have

$$
\begin{aligned}
r(M) &+ \sum_{k=1}^{n} r(q_{n+1,k}(M)) \\
&= \left[r(R \times K) + \sum_{k=1}^{n} r(q_{n+1,k}(R \times K)) \right] \\
&\quad + \sum_{j=1}^{m} \left[r(\{p_j\} \times G_j) + \sum_{k=1}^{n} r(q_{n+1,k}(\{p_j\} \times G_j)) \right] \\
&\quad + \left[r(F) + \sum_{k=1}^{n} r(q_{n+1,k}(F)) \right]
\end{aligned}
$$

We have $r(R \times K) = 0$, $q_{n+1,n}(R \times K) = R$, $q_{n+1,k}(R \times K) = q_{n,k}(R)$ for $k = 1, \ldots, n-1$. As $r \in \mathscr{L}_n$, we have by the induction assumption $\sum_{k=1}^{n} r(q_{n+1,k}(R \times K)) = 0$. Further, $r(F) = 0$ by (1) and $q_{n+1,k}(F) = \varnothing$ for $k = 1, \ldots, n$. As $q_{n+1,n}(\{p_j\} \times G_j) = \{p_j\}$ and $q_{n+1,k}(\{p_j\} \times G_j) = \varnothing$ for $k = 1, \ldots, n-1$, it is sufficient to prove $r(\{p_j\} \times G_j) + r(\{p_j\}) = 0$ for $j = 1, \ldots, m$.

If $p_j \in Q_n$, we have $r(\{p_j\}) = 1$ and

$$
\begin{aligned}
r(\{p_j\} \times G_j) &= f_{\{p_j\} \times G_j}(p_j) \\
&= |[\{p_j\} \times (K - G_j)] \cap Q'_{n+1}| - |[\{p_j\} \times (K - G_j)] \cap Q_{n+1}| \\
&= |(K - G_j) \cap K_2| - |(K - G_j) \cap K_1| \\
&= |K_2 - G_j| - |K_1 - G_j| = -1
\end{aligned}
$$

by (1). If $p_j \in Q'_n$, we have analogously $r(\{p_j\}) = -1$ and $r(\{p_j\} \times G_j) = 1$. In both cases $r(\{p_j\}) + r(\{p_j\} \times G_j) = 0$, so that statement (c) holds for all $M \in \mathscr{L}_{n+1}$.

IX. $\langle X, \mathscr{L} \rangle$ is not a Boolean algebra.

Clearly

$$
A = \{x_1, x'_1\} + \prod_{i=2}^{\infty} K \in \mathscr{L} \qquad \text{and} \qquad B = \{x_1, x'_2\} \times \prod_{i=2}^{\infty} K \in \mathscr{L}
$$

where $x_1 \in K_1$, $x'_1, x'_2 \in K_2$, and $x'_1 \neq x'_2$. We prove that

$$
A \cap B = \{x_1\} \times \prod_{i=2}^{\infty} K
$$

does not belong to $\mathscr{L}$. Suppose on the contrary that $\{x_1\} \times \prod_{i=2}^{n} K \in \mathscr{L}_n$ for some n. Then $n \geq 2$,

$$r\left(\{x_1\} \times \prod_{i=2}^{n} K\right) = 0,$$

$$r\left(q_{n,k}\left(\{x_1\} \times \prod_{i=2}^{n} K\right)\right) = r\left(\{x_1\} \times \prod_{i=2}^{k} K\right) = 0$$

for $k = 2, 3, \ldots, n-1$ and

$$r\left(q_{n,1}\left(\{x_1\} \times \prod_{i=2}^{n} K\right)\right) = r(\{x_1\}) = 1.$$

So condition 3 of VIII is not satisfied and $A \cap B \notin \mathscr{L}$.

Remark. The logic $\langle X, \mathscr{L} \rangle$ constructed in Theorem 1 has cardinality ω_1. By Müller *et al.* (1993), Theorem 4.1, it is not possible to construct a countable example with these properties.

In the present construction we used essentially the fact that the intersection of any two elements of the logic is equal to the union of two other elements. By Müller *et al.* (1993), a concrete σ-logic (i.e., a logic closed under taking countable disjoint unions) with this property is already a Boolean algebra. We conjecture that a concrete σ-logic such that every (finite additive) state on it is Jauch–Piron is necessarily a Boolean algebra (however, it may seem unnatural to consider σ-logics and finite-additive states). The situation is much better in the class of concrete complete logics (i.e., concrete logics closed under taking arbitrary disjoint unions). The following theorem gives several equivalent conditions for complete concrete logics with all states Jauch–Piron. The results confirm also the connection between the Jauch–Piron property of states and the covering properties studied in Müller *et al.* (1993).

Theorem 2. Let $\langle X, \mathscr{L} \rangle$ be a complete concrete logic. Then the following statements are equivalent:

1. $\langle X, \mathscr{L} \rangle$ is a complete Boolean algebra.
2. Every state on $\langle X, \mathscr{L} \rangle$ is Jauch–Piron.
3. Every two-valued state on $\langle X, \mathscr{L} \rangle$ carried by a point of X is Jauch–Piron.
4. $\bigcup \{M \in \mathscr{L}, M \subset A \cap B\} = A \cap B$ for every $A, B \in \mathscr{L}$.

[I.e., $\langle X, \mathscr{L} \rangle$ is of class $C_{\mathrm{Card}\,\mathscr{L}}$ in the notation of Müller *et al.* (1993).]

Proof. The implications $1 \Rightarrow 2 \Rightarrow 3$ are clear.

$3 \Rightarrow 4$ (Müller *et al.*, 1993): Let $A, B \in \mathscr{L}$ and

$$\bigcup \{M \in \mathscr{L}, M \subset A \cap B\} \neq A \cap B$$

Take $x \in (A \cap B) - \bigcup \{M \in \mathscr{L}, M \subset A \cap B\}$ and let s be the two-valued state carried by x. Then s is not Jauch–Piron.

$4 \Rightarrow 1$: We shall use the following convention:

Let α be an ordinal number and let $\mathscr{M} = \{M_\beta\}_{\beta < \alpha}$ be a system of sets. We say that $\mathscr{M}$ is increasing if $M_{\beta_1} \subset M_{\beta_2}$ for $\beta_1 < \beta_2$. We say that $\mathscr{M}$ is decreasing if $M_{\beta_1} \supset M_{\beta_2}$ for $\beta_1 < \beta_2$.

The proof will be carried out in several steps:

(a) Let α be an ordinal number and let M_β $(\beta < \alpha)$ be an increasing system of sets of $\mathscr{L}$. Then $\bigcup_{\beta < \alpha} M_\beta \in \mathscr{L}$.

Proof. For $\beta \leq \alpha$, denote $K_\beta = \bigcup_{\gamma < \beta} M_\gamma$. We prove that $K_\beta \in \mathscr{L}$ for every $\beta \leq \alpha$. Suppose on the contrary that $K_{\beta_0} \notin \mathscr{L}$ and let β_0 be the smallest ordinal with this property. For every $x \in K_{\beta_0} = \bigcup_{\gamma < \beta_0} M_\gamma$ there exists the smallest ordinal γ_x such that $x \in M_{\gamma_x}$. Thus, K_{β_0} can be written as a disjoint union $K_{\beta_0} = \bigcup_{\gamma < \beta_0} [M_\gamma - K_\gamma]$. By the assumption, $K_\gamma \in \mathscr{L}$ for all $\gamma < \beta_0$, hence $K_{\beta_0} \in \mathscr{L}$, a contradiction.

(b) Let α be an ordinal number and let M_β $(\beta < \alpha)$ be a decreasing system of sets of $\mathscr{L}$. Then $\bigcap_{\beta < \alpha} M_\beta \in \mathscr{L}$.

Proof. We have $\bigcap_{\beta < \alpha} M_\beta = X - \bigcup_{\beta < \alpha} (X - M_\beta)$, where $X - M_\beta$ is an increasing system. Thus $\bigcap_{\beta < \alpha} M_\beta \in \mathscr{L}$.

(c) Let α be an ordinal number and let $M_\beta \in \mathscr{L}$ $(\beta < \alpha)$. Let $x \in \bigcap_{\beta < \alpha} M_\beta$. Then there exists $K \in \mathscr{L}$ such that $x \in K$ and $K \subset \bigcap_{\beta < \alpha} M_\beta$.

Proof. By transfinite induction we define a decreasing system $K_\beta \in \mathscr{L}$ $(\beta < \alpha)$ of sets containing x. Set $K_0 = M_0$.

If β is a nonlimit ordinal, $\beta = \gamma + 1$, then $K_\gamma \in \mathscr{L}$, $M_\gamma \in \mathscr{L}$, $x \in K_\gamma \cap M_\gamma$. Therefore $K_\gamma \cap M_\gamma = \bigcup \{L \in \mathscr{L}, L \subset K_\gamma \cap M_\gamma\}$ and there exists a set $K_\beta \in \mathscr{L}$, $K_\beta \subset K_\gamma \cap M_\gamma$, $x \in K_\beta$.

If β is a limit ordinal, define $K_\beta = \bigcap_{\gamma < \beta} K_\gamma$.

Clearly K_β $(\beta < \alpha)$ is a decreasing system of sets of $\mathscr{L}$, $x \in K_\beta$ for all $\beta < \alpha$, and $K_\beta \subset M_\gamma$ for all $\gamma < \beta$. Hence $K = \bigcap_{\beta < \alpha} K_\beta$ satisfies the required conditions.

Notation. Let $\mathscr{M}, \mathscr{K} \subset \mathscr{L}$ be two systems of subsets of X. We shall write $\mathscr{K} \prec \mathscr{M}$ if $\bigcup \mathscr{K} = \bigcup \mathscr{M}$ and $K \subset M$ for every $K \in \mathscr{K}$ and $M \in \mathscr{M}$ such that $K \cap M \neq \varnothing$.

It is easy to see that the relation $\prec$ is transitive.

(d) Let $\mathcal{M}=\{M_i, i\in I\}\subset\mathcal{L}$ be a system of subsets of X. Then there exists a system $\mathcal{K}\subset\mathcal{L}$ such that $\mathcal{K}\prec\mathcal{M}$.

Proof. Let $x\in\bigcup\mathcal{M}$. Denote $I_1=\{i\in I, x\in M_i\}$, $I_2=\{i\in I, x\notin M_i\}$. Apply the previous statement to the system $\{M_i, i\in I_1\}\cup\{X-M_i, i\in I_2\}$. There exists a set $K_x\in\mathcal{L}$, $x\in K_x$, $K_x\subset\bigcap_{i\in I_1} M_i\cap\bigcap_{i\in I_2}(X-M_i)$. It is easy to see that the system $\mathcal{K}=\{K\in\mathcal{L}, K=K_x$ for some $x\in\bigcup\mathcal{M}\}$ has the required property.

(e) Let $A, B\in\mathcal{L}$. Then $A\cap B\in\mathcal{L}$.

Proof. Set $\mathcal{K}_0=\{M\in\mathcal{L}, M\subset A\cap B\}$. By the assumption, $\bigcup\mathcal{K}_0=A\cap B$. For every positive integer n we define a system $\mathcal{K}_n\subset\mathcal{L}$ such that $\bigcup\mathcal{K}_n=\mathrm{A}\cap\mathrm{B}$ $\mathcal{K}_n\prec\mathcal{K}_{n-1}$. Let $x\in A\cap B$. For every n we find a set $K_n\in\mathcal{K}_n$ with $x\in K_n$. This means that $K_{n+1}\subset K_n$ for every n. Set $K_x=\bigcap_{n=0}^{\infty}K_n\in\mathcal{L}$. The set K_x is independent of the choice of the sequence K_n. Indeed, suppose $K_n'\in\mathcal{K}_n$, $x\in K_n'$ $(n=0, 1, \ldots)$. As $K_n\cap K_{n+1}'\neq\emptyset$, we have $K_{n+1}'\subset K_n$ and $\bigcap_{n=0}^{\infty}K_n'\subset\bigcap_{n=0}^{\infty}K_n$. The opposite inclusion can be obtained analogously.

Suppose $K_x\cap K_y\neq\emptyset$ for two elements $x, y\in A\cap B$, $x\neq y$. Similarly as above we get $K_x=K_y$, so that $\mathcal{K}=\{K\in\mathcal{L}, K=K_x$ for some $x\in A\cap B\}$ is a disjoint system. Hence $A\cap B=\bigcup\mathcal{K}\in\mathcal{L}$.

(f) $\langle X, \mathcal{L}\rangle$ is a complete Boolean algebra.

Proof. Let α be an ordinal number and let $M_\beta\in\mathcal{L}$ $(\beta<\alpha)$. For $\beta\leq\alpha$ denote $K_\beta=\bigcap_{\gamma<\beta}M_\gamma$. Clearly K_β $(\beta\leq\alpha)$ is a decreasing system of sets. We prove by transfinite induction that $K_\beta\in\mathcal{L}$ for every $\beta\leq\alpha$. For $\beta=0$ we have $K_0=X\in\mathcal{L}$. For nonlimit ordinals we have $K_{\beta+1}=K_\beta\cap M_\beta\in\mathcal{L}$ by (e). If β is a limit ordinal we have $K_\beta=\bigcap_{\gamma<\beta}K_\gamma\in\mathcal{L}$ by (b). This finishes the proof.

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